Orthogonal subsets of root systems and the orbit method

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0. Introduction and statement of the main result

1. Introduction

1.1. The main tool in studying irreducible complex representations of finite unipotent groups is the *orbit method*. It was created by A.A. Kirillov for nilpotent Lie groups over $\mathbb{R}$ [12], [13], and then adapted by D. Kazhdan for finite groups [11] (see also [14] and the paper [3], where the theory of ℓ-adic sheaves for unipotent groups is explained). Here we consider the groups $U(q)$ and U, the maximal unipotent subgroups of Chevalley groups over a finite field $\mathbb{F}_q$ and its algebraic closure respectively.

The orbit method establishes a bijection between the set of equivalence classes of irreducible representations of $U(q)$ and the set of orbits of the *coadjoint representation* of $U(q)$. Further, a lot of questions about representations can be interpreted in terms of orbits. Note that the problem of complete description of orbits remains unsolved and seems to be *very* difficult. On the other hand, a lot of information about some special types of orbits, representations and characters is known.

In particular, a description of *regular* orbits (i.e., orbits of maximal dimension) of the group UT_n of all unipotent triangular matrices of size $n \times n$ is known [13]. *Subregular* orbits (i.e., orbits of second maximal dimension) and corresponding characters[1] were described in [7] and [8]. As a generalization, A.N. Panov considered orbits of the group UT_n associated with involutions in the symmetric group. In [16], he obtained a formula for the dimension of such an orbit.

It's well-known that the group UT_n corresponds to the root system of type A_{n-1}. In order to generalize the results of A. N. Panov, we introduced the concept of orbits associated with orthogonal subsets of root systems. In the paper [10], we studied these orbits for the case of classical root systems. For orthogonal subsets of special kind of the root systems of types B_n and D_n, we also obtained a formula involving the corresponding irreducible characters, see [9, Theorem 3.8]).

The main goal of this paper is to generalize the results of [10] to the general case of an *arbitrary* root system, not only the classical one. The structure of the paper ia as follows. In the remainder of this Section, we give necessary definitions and formulate the main result (see Theorem 1.2). In Section 1, we prove some preliminary technical Lemmas and consider some important examples. In Section 2, we prove the Main Theorem for simply laced root systems (see Propositions 3.4 and 3.5). In Section 3, we prove the Main Theorem for multiply laced root systems.

The author is sincerely grateful to his scientific advisor professor A.N. Panov for constant attention to this work.

1.2. In this Subsection, we shall briefly recall some basic facts concerning Chevalley groups over finite fields. We also give some definitions, which are needed to formulate the main result.

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[1]The description of the irreducible character corresponding to a given orbit is itself a non-trivial problem, see, e.g., the papers of C.A.M. Andrè and A. Neto [1], [2] for the description of the so-called *super-characters*. In this paper, we concentrate on orbits, not on characters.

Let Φ be a reduced root system, $\Delta \subset \Phi$ a subset of fundamental roots, Φ^+ and Φ^- the corresponding subsets of positive and negative roots respectively (see [4]). As usual, we denote by $W = W(\Phi)$ the Weyl group of the root system Φ. Let $r_\alpha \in W$ be the reflection on the hyperplane orthogonal to a given root $\alpha \in \Phi$.

Let p be a prime, $\mathbb{F}_q$ the field with $q = p^r$ elements for some $r \geqslant 1$, $k = \overline{\mathbb{F}}_q$ its algebraic closure. Let $G(q) = G_{\mathrm{sc}}(\Phi, \mathbb{F}_q)$ (resp. $G = G_{\mathrm{sc}}(\Phi, k)$) be the simply connected *Chevalley group* over the field $\mathbb{F}_q$ (resp. over k) with the root system Φ (see the classical book [18] for precise definitions; see also [17]). Recall that there exists a so-called *Chevalley basis* of the Lie algebra $\mathfrak{g}$ of the group G. In particular, this basis contains the root vectors $\{e_\alpha,\ \alpha \in \Phi^+\}$ satisfying $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$, where $N_{\alpha\beta}$ are the so-called *Chevalley structure constants* (here we set $N_{\alpha\beta} = 0$ if $\alpha + \beta \notin \Phi$).

The subspace $\mathfrak{u} = \sum_{\alpha \in \Phi^+} k e_\alpha$ is a nilpotent Lie subalgebra of $\mathfrak{g}$. We assume from now on that p is not less than the Coxeter number of the root system Φ. This implies $[x_1, [x_2, [\ldots, [x_{p-1}, x_p] \ldots]] = 0$ for all $x_i \in \mathfrak{u}$, so the orbit method applies [3, Theorem 2.2 and §3.3].

Since p is sufficiently large, the exponential map $\exp\colon \mathfrak{u} \to G$ is well-defined. Its image U is a maximal unipotent subgroup of G and the map $\exp\colon \mathfrak{u} \to U$ is a bijection. Further, U is generated as a subgroup of G by all root subgroups corresponding to positive roots from Φ, and $\mathfrak{u}$ is the Lie algebra of the group U.

Thus, the group U acts on $\mathfrak{u}$ via the adjoint representation. The dual representation of U in the space $\mathfrak{u}^*$ of all k-linear functions on $\mathfrak{u}$ is called *coadjoint*. One can see that the coadjoint action has the form

$$\exp(y).f(x) = f(\exp \mathrm{ad}_{-y} x), \quad x, y \in \mathfrak{u}, f \in \mathfrak{u}^*.$$

Here $\mathrm{ad}_y x = [y, x]$; since ad_y is a nilpotent linear operator on $\mathfrak{u}$, the map $\exp \mathrm{ad}_y = \sum_{i=0}^{\infty} \mathrm{ad}_y^i / i!\colon \mathfrak{u} \to U$ is well-defined.

One can define the algebra $\mathfrak{u}(q) \subset \mathfrak{g}(q)$, the group $U(q)$ and its coadjoint representation in the space $\mathfrak{u}^*(q) = (\mathfrak{u}(q))^*$ by the similar way. Let us fix an embedding $\mathbb{F}_q \subset k$. Then $\mathfrak{u}(q)$ can be canonically embedded in $\mathfrak{u}$. In the paper we concentrate[2] on orbits of elements from $\mathfrak{u}(q)$ under the coadjoint action of the group U, not of the group $U(q)$.

Now we shall give the main definition. Let D be a subset of Φ^+ consisting of pairwise orthogonal roots, then D is called *orthogonal*. Let $\xi = (\xi_\beta)_{\beta \in D}$ be a set of non-zero scalars from k. Denote by $\{e_\alpha^*\}$ the basis of $\mathfrak{u}^*$ dual to the basis $\{e_\alpha,\ \alpha \in \Phi^+\}$ of the algebra $\mathfrak{u}$. Set

$$f = f_{D,\xi} = \sum_{\beta \in D} \xi_\beta e_\beta^* \in \mathfrak{u}^*.$$

Definition 1.1. We say that the orbit $\Omega = \Omega_{D,\xi} \subset \mathfrak{u}^*$ of the element f under the coadjoint action of the group U is *associated* with the subset D. The element f is called the *canonical form* on the orbit Ω.

Note that many important examples deal with orbits associated with orthogonal subsets, see Subsection 2.4.

1.3. To formulate the main result, we need some facts concerning involutions in the Weyl group of the root system Φ. Namely, for a given orthogonal subset $D \subset \Phi^+$, we put

$$\sigma = \sigma_D = \prod_{\beta \in D} r_\beta \in W$$

(commuting reflections r_β are taken in any fixed order). Obviously, σ is an involution, i.e., an element of order two of the group W.

[2]The set of irreducible representations of the group $U(q)$ is in bijection with the set of coadjoint orbits of $U(q)$, but a lot of questions about representations can be interpreted in terms of orbits of the group U [11].

To each element $w \in W$ one can assign the numbers $l(w)$ and $s(w)$. By definition, $l(w)$ (resp. $s(w)$) is the length of a reduced (the shortest) expression of w as a product of simple (resp. arbitrary) reflections. One has $s(\sigma) = |D|$. It's well-known that $l(\sigma) = |\Phi_\sigma|$, where $\Phi_\sigma = \{\alpha \in \Phi^+ \mid \sigma\alpha < 0\}$. As usual, $\alpha > 0$ means that $\alpha \in \Phi^+$, and $\alpha < 0$ means that $\alpha \in \Phi^-$. Furthermore, by $<$ we denote the usual partial order on Φ: by definition, $\alpha > \beta$ (or $\beta < \alpha$) if $\alpha - \beta$ is a sum of positive roots.

Things now are ready to formulate the Main Theorem. Since Ω is an irreducible affine variety (see [5, Proposition 8.2] and [19, Proposition 2.5]), one can ask how to compute $\dim\Omega$, the dimension of Ω over k. (In fact, if f is an element of $\mathfrak{u}^*(q)$ and $\Omega(q), \Omega$ are its orbits under the action of the groups $U(q)$, U respectively, then the complex dimension of the irreducible representation of $U(q)$ corresponding to the orbit $\Omega(q)$ is equal to $q^{\dim\Omega/2}$, see [11].)

Theorem 1.2. *Let D be an orthogonal subset of Φ^+, ξ a set of non-zero scalars from k and $\Omega = \Omega_{D,\xi}$ the orbit associated with D. Then $\dim\Omega$ does not depend on ξ and is less or equal to $l(\sigma) - s(\sigma)$.*

Remark 1.3. i) This Theorem proves Conjecture 1.4 from [10]. Note that in many cases (e.g., for elementary orbits) $\dim\Omega$ is *equal* to $l(\sigma) - s(\sigma)$, see Subsection 2.4.

ii) On the other hand, for classical groups, the difference between $\dim\Omega$ and $l(\sigma) - s(\sigma)$ can be computed explicitly; furthermore, a polarization of $\mathfrak{u}$ at the canonical form on Ω can be constructed, see [10, Theorems 1.1 and 1.2] (polarizations play an important role in the explicit construction of the representation corresponding to a given orbit). We don't know how to do this for an arbitrary root system.

2. Lemmas and examples

2.1. Without loss of generality we can assume Φ to be an irreducible root system. Indeed, let $\Phi = \bigcup_{i=1}^m \Phi_i$ be the decomposition of Φ into the union of its pairwise orthogonal irreducible components. Put $D = \bigcup_{i=1}^m D_i$, where $D_i = D \cap \Phi_i$. Put also $\mathfrak{u}_i = \sum_{\alpha\in\Phi_i^+} ke_\alpha$ for all i, and let $\mathfrak{u}_i^*$ be the subspace of $\mathfrak{u}^*$ dual to the subalgebra $\mathfrak{u}_i$. Denote by $f_i \in \mathfrak{u}_i^*$ the restriction of f to $\mathfrak{u}_i$. Denote also by $\Omega_i \subset \mathfrak{u}_i^*$ the orbit of f_i under the coadjoint action of the group $U_i = \exp(\mathfrak{u}_i)$. Finally, denote by W_i the Weyl group of the root system Φ_i and put $\sigma_i = \sigma_{D_i} \in W_i$.

Lemma 2.1. *Suppose that Theorem* 1.2 *holds for all Ω_i, $i = 1, \ldots, m$. Then Theorem* 1.2 *holds for the orbit Ω.*

Proof. Obviously, if $\alpha \in \Phi_i$, $\beta \in \Phi_j$ and $i \neq j$, then $\alpha+\beta \notin \Phi$. Hence if $x = \sum_{i=1}^m x_i$, $y = \sum_{j=1}^m y_j$, $x_i, y_i \in \mathfrak{u}_i$, then $\mathrm{ad}_{-y_j}^r x_i = 0$ for all $j \neq i$, $r > 0$. Since $\exp\mathrm{ad}_{-y_i}x_i \in \mathfrak{u}_i$ for all i, we obtain

$$\begin{aligned}\exp(y).f(x) &= \left(\sum_{i=1}^m f_i\right)\left(\sum_{j=1}^m \exp\mathrm{ad}_{-y_j}x_j\right)\\ &= \sum_{i=1}^m f_i(\exp\mathrm{ad}_{-y_i}x_i) = \sum_{i=1}^m \exp(y_i).f_i(x_i),\end{aligned}$$

so the maps

$$\begin{aligned}&\Omega_1 \times \ldots \times \Omega_m \to \Omega\colon (\lambda_1, \ldots, \lambda_m) \mapsto \lambda_1 + \ldots + \lambda_m \text{ and}\\ &\Omega \to \Omega_1 \times \ldots \times \Omega_m\colon \lambda \mapsto (\lambda|_{\mathfrak{u}_1}, \ldots, \lambda|_{\mathfrak{u}_m})\end{aligned}$$

are isomorphisms of affine varieties inverse to each other.

Suppose that Theorem 1.2 holds for all Ω_i. Let $\xi_i = (\xi_\beta)_{\beta\in D_i}$. Since Ω_i coincides with Ω_{D_i,ξ_i}, $\dim\Omega$ doesn't depend on ξ. On the other hand, if $i \neq j$, then $r_\beta|_{\Phi_j} = \mathrm{id}_{\Phi_j}$ for all $\beta \in D_i$, so $l(\sigma) = \sum_{i=1}^m l(\sigma_i)$. Finally, $s(\sigma) = |D| = |\bigcup_{i=1}^m D_i| = \sum_{i=1}^m |D_i| = \sum_{i=1}^m s(\sigma_i)$. This concludes the proof. □

From now on and to the end of the paper, we assume Φ to be irreducible.

2.2. Sometimes orbits associated with different orthogonal subsets coincide. To give the precise statement, we need to introduce the important concept of singular roots.

Definition 2.2. Let $\beta \in \Phi^+$ be a positive root. Roots $\alpha, \gamma \in \Phi^+$ are called *β-singular* if $\alpha+\gamma=\beta$. The set of all β-singular roots is denoted by $S(\beta)$.

Of course, one can easily describe the set of all β-singular roots for a given root β (see [10, formula (2)] for the case of classical groups).

Suppose that there exist $\beta_0, \beta_1 \in D$ such that $\beta_0 \in S(\beta_1)$. Put $D' = D \setminus \{\beta_0\}$, $\xi' = \xi \setminus \{\xi_{\beta_0}\}$, $f' = f_{D',\xi'}$. Let Ω' be the orbit of f'.

Lemma 2.3. *The orbit Ω coincide*[3] *with the orbit Ω'.*

Proof. Suppose $\beta_1 = \beta_0 + \alpha$. Then $\|\alpha\|^2$ is equal to $\|\beta_0\|^2 + \|\beta_1\|$. Since Φ is irreducible, we conclude that Φ is multiply laced (the root α is long, the roots β_0, β_1 are short); further, the square of the length of a long root is twice to the square of the length of a short one. (In other words, the root system Φ is of type B_n, C_n or F_4.)

Set $\widetilde{f} = \exp(ce_\alpha).f'$ for some $c \in k^*$. One has

$$\widetilde{f}(e_\gamma) = f'(e_\gamma) - c \cdot f'(\mathrm{ad}_{e_\alpha} e_\gamma) + \frac{1}{2}c^2 \cdot f'(\mathrm{ad}^2_{e_\alpha} e_\gamma) - \ldots$$

for a given root $\gamma \in \Phi^+$. Suppose $\widetilde{f}(e_\gamma) \neq 0$, then there exists $N \geqslant 0$ such that $\gamma + N\alpha \in D'$. Of course, this holds for $\gamma = \beta_0$ and $N = 1$, because $\beta_0 + \alpha = \beta_1 \in D'$. Suppose that $N \geqslant 2$ and $\beta_0 + N\alpha = \beta \in D'$ (and so $\beta \neq \beta_1$). In this case $8||\beta_0||^2 \leqslant N^2 \cdot ||\alpha||^2 = ||\beta_0||^2 + ||\beta||^2$, a contradiction. Hence $\widetilde{f}(e_{\beta_0}) = -c \cdot N_{\alpha\beta_0} \cdot \xi_{\beta_1}$.

Suppose now that $\gamma \neq \beta_0$ and $\gamma + N\alpha = \beta \in D'$. If $N = 0$, then $\gamma \in D'$, so we can assume $N \geqslant 1$. We see that $\gamma + (N-1)\alpha = \beta - \alpha = \beta - \beta_1 + \beta_0$. If $N = 1$, then $\beta \neq \beta_1$, so $||\gamma||^2 = ||\beta + \beta_0||^2 + ||\beta_1||^2$. But $||\beta_1||^2 = ||\beta_0||^2$ (β_1 and β_0 are short), and $||\beta + \beta_0||^2 \geqslant 2||\beta_0||^2$ (the roots β, β_0 are either equal or orthogonal). Thus, $||\gamma||^2 \geqslant 3||\beta_0||^2$, a contradiction. Hence $N \geqslant 2$. On the other hand, $(\beta, \alpha) = (\beta, \beta_1) - (\beta, \beta_0)$ and $\gamma = \beta - N\alpha$. If $\beta \neq \beta_1$, then $(\beta, \alpha) \leqslant 0$, so $||\gamma||^2 = ||\beta||^2 + N^2 \cdot ||\alpha||^2 - 2N \cdot (\beta, \alpha) \geqslant 4||\alpha^2||$, a contradiction. But if $\beta = \beta_1$, then $||\gamma||^2 = ||N\beta_0 + (1-N)\beta_1||^2 = (N^2 + (1-N)^2)||\beta_0||^2 \geqslant 5||\beta_0||^2$, a contradiction.

We conclude that $\widetilde{f}(e_\gamma) = f'(e_\gamma) = f(e_\gamma)$ for all $\gamma \neq \beta_0$. Hence if $c = -\xi_{\beta_0}/(N_{\alpha\beta_0} \cdot \xi_{\beta_1})$, then $\widetilde{f}$ coincides with f. By definition, $\widetilde{f} \in \Omega'$, so $\Omega' = \Omega$ as required. □

From now on and to the end of the paper, we assume that $S(\beta) \cap D = \varnothing$ for all $\beta \in D$.

2.3. To prove the Main Theorem for simply laced root systems, we need some more preparations. Let η, η', η_i, θ, θ', θ_j, ψ, ψ', ψ_l, ψ'_l be distinct positive roots; assume the roots η, η', η_i to be pairwise orthogonal and assume the root η to be maximal among all η's w.r.t the usual order on Φ. Consider the following cases:

$$\begin{array}{llllll}
1. & \eta = \theta + \psi = \theta' + \psi', & 2. & \eta = \theta + \psi, & 3. & \eta = \theta + \psi, \\
 & \eta_1 = \theta_1 + \psi, & & \eta' = \theta' + \psi, & & \eta_1 = \theta_1 + \psi, \\
 & \eta_2 = \theta_2 + \psi, & & \eta_1 = \theta + \psi_1, & & \eta_2 = \theta + \psi_2, \\
 & \eta' = \theta_1 + \psi', & & \eta_2 = \theta + \psi_2, & & \eta_3 = \theta_1 + \psi_3,
\end{array}$$

$$\begin{array}{ll}
4. & \eta = \theta + \psi = \theta'_1 + \psi'_1, \\
 & \eta_1 = \theta_1 + \psi = \theta'_1 + \psi', \\
 & \eta' = \theta + \psi' = \theta_1 + \psi'_1, \\
 & \{\theta, \theta_1, \theta'_1, \psi, \psi', \psi'_1\} \cap S(\eta_2) \neq \varnothing.
\end{array}$$

[3] Cf. [10, Proposition 2.1].

Definition 2.4. A set of positive roots satisfying the conditions of type 1, 2, 3 or 4 is called *non-admissible.*

Lemma 2.5. *There are no non-admissible subsets in* D_5^+*.*

Proof. Straightforward. □

2.4. Before the proof of the Main Theorem, let us consider some examples of orbits associated with orthogonal subsets. Let us firstly consider the case $\Phi = A_{n-1}$ (i.e., $U = \mathrm{UT}_n$, the unitriangular group). It's convenient to identify A_{n-1}^+ with the subset of $\mathbb{R}^n$ of the from $\{\varepsilon_i - \varepsilon_j, 1 \leqslant i < j \leqslant n\}$ (by $\{\varepsilon_i\}_{i=1}^n$ we denote the standard basis of $\mathbb{R}^n$). The Weyl group of A_{n-1} is isomorphic to S_n, the symmetric group on n letters.

Example 2.6. Let $D = D_{\mathrm{reg}} = \{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}, \ldots, \varepsilon_{n_1} - \varepsilon_{n_2}\}$, where $n_1 = [n/2]$, $n_2 = n - n_1 + 1$. Then σ is the longest element of the Weyl group W and $\Phi_\sigma = \Phi^+$ (i.e., $\sigma(\alpha) < 0$ for *all* positive roots α). Then the orbit Ω is *regular*, i.e., has the maximal dimension among all coadjoint orbits. The dimension of Ω equals

$$\dim \Omega = l(\sigma) - s(\sigma) = 2\mu(n),\ \mu(n) = (n-2) + (n-4) + \ldots$$

ii) Now let

$$D = D_{\mathrm{sreg}} = (D_{\mathrm{reg}} \setminus \{\varepsilon_i - \varepsilon_{n-i+1}, \varepsilon_{i+1} - \varepsilon_{n-i}\}) \cup \{\varepsilon_i - \varepsilon_{n-i}, \varepsilon_{i+1} - \varepsilon_{n-i+1}\}.$$

In this case, the orbit Ω is *subregular*, i.e., has the second maximal dimension $\dim \Omega = l(\sigma) - s(\sigma) = 2\mu(n) - 2$, see [7, Section 3].

Example 2.7. Let Φ be an arbitrary root system. Suppose that $|D| = 1$. Then the orbit Ω is called *elementary.* It's easy to see that $\dim \Omega = |S(\beta)|$ [15, Section 4]. It's straightforward to check that $l(\sigma) - s(\sigma)$ coincides with $|S(\beta)|$ (see [10, Section 4] for the case of classical groups).

Example 2.8. On the other hand, if $\Phi = B_3$ (recall that $B_3^+ = \{\varepsilon_i,\ \varepsilon_i \pm \varepsilon_j,\ 1 \leqslant i < j \leqslant 3\}$) and $D = \{\varepsilon_1, \varepsilon_2 + \varepsilon_3\}$, then the dimension of Ω is *less* than $l(\sigma) - s(\sigma)$, because $\dim \Omega = 4$ and $l(\sigma) - s(\sigma) = 6$ (see [9] or [10]).

3. Simply laced root systems

3.1. Throughout this Section, Φ is a *simply laced* root system, i.e., all roots from Φ have the same length. (In other words, Φ is of type A_n, D_n, E_6, E_7 or E_8). Without loss of generality, suppose that the length of a root from Φ equals 1. Then the inner product of two non-orthogonal roots from Φ equals either ± 1 or $\pm 1/2$. Moreover, suppose $\alpha, \beta \in \Phi^+$, then $(\beta, \alpha) = 1/2$ if and only if either $\alpha \in S(\beta)$ or $\beta \in S(\alpha)$; in this case, $r_\beta \alpha = \alpha - \beta$. On the other hand, $(\alpha, \beta) = -1/2$ if and only if $\alpha + \beta \in \Phi^+$; in this case, $r_\beta \alpha = \alpha + \beta$.

As above, let D be an orthogonal subset of Φ^+, ξ a set of non-zero scalars from k, $\Omega = \Omega_{D,\xi}$ the associated coadjoint orbit and f the canonical form on Ω. Firstly, let us prove that the dimension of the orbit Ω is less or equal to$l(\sigma) - s(\sigma)$. The proof is by induction on the rank of Φ. The base ($\mathrm{rk}\,\Phi = 1$, i.e., $\Phi = A_1$) is straightforward. To perform the inductive step, it's enough to prove the statement only for irreducible root systems of a given rank, as shows Lemma 2.1.

For the case $|D| = 1$ (i.e., the case of elementary orbits), there is nothing to prove, see Example 2.7. Suppose $|D| > 1$. Pick a root β maximal among all roots from D. Put $\widetilde{D} = D \setminus \{\beta\}$. In order to use the inductive hypothesis, we'll define the root system of rank less than the rank of Φ. Precisely, put $\mathcal{A} = \{\alpha \in \Phi^+ \mid (\alpha, \beta) \neq 0\}$ and $\widetilde{\Phi} = \pm\widetilde{\Phi}^+$, where

$$\widetilde{\Phi}^+ = \Phi^+ \setminus \mathcal{A} = \{\alpha \in \Phi^+ \mid (\alpha, \beta) = 0\}.$$

The following Lemma is obvious.

Lemma 3.1. *The set $\widetilde{\Phi}$ is a root system.* $\square$

By construction, $\operatorname{rk}\widetilde{\Phi} < \operatorname{rk}\Phi$. Obviously, $\widetilde{D} = D \cap \widetilde{\Phi}^+$. Denote by $\widetilde{\mathfrak{u}}$ the subalgebra of $\mathfrak{u}$ spanned by all vectors e_α, $\alpha \in \widetilde{\Phi}^+$. Put $\widetilde{f} = f|_{\widetilde{\mathfrak{u}}} \in \widetilde{\mathfrak{u}}^* \subset \mathfrak{u}^*$, so $f = \xi_\beta e_\beta^* + \widetilde{f}$. Put also $\widetilde{U} = \exp(\widetilde{\mathfrak{u}})$, $\widetilde{\xi} = \xi \setminus \{\xi_\beta\}$. Let $\widetilde{\Omega} = \Omega_{\widetilde{D},\widetilde{\xi}} \subset \widetilde{\mathfrak{u}}^*$ be the coadjoint orbit of the group $\widetilde{U}$ associated with $\widetilde{D}$. Then $\widetilde{f}$ is the canonical form on the orbit $\widetilde{\Omega}$.

Finally, let $\widetilde{\sigma}$ be the involution in the Weyl group $\widetilde{W}$ of the root system $\widetilde{\Phi}$ corresponding to the subset $\widetilde{D}$. By the inductive assumption, $\dim\widetilde{\Omega}$ is less or equal to $l(\widetilde{\sigma}) - s(\widetilde{\sigma})$. Obviously, $s(\sigma) = s(\widetilde{\sigma}) + 1$, so it remains to compare $l(\sigma)$ with $l(\widetilde{\sigma})$.

3.2. Let $\mathfrak{a}$ be the *radical* of the bilinear form $x, y \mapsto f([x,y])$, $x, y \in \mathfrak{u}$, i.e.,

$$\mathfrak{a} = \mathfrak{rad}_{\mathfrak{u}} f = \{x \in \mathfrak{u} \mid f([x,y]) = 0 \text{ for all } y \in \mathfrak{u}\}.$$

It's well-known that $\dim\Omega = \operatorname{codim}_{\mathfrak{u}}\mathfrak{a} = |\Phi^+| - \dim\mathfrak{a}$ [2, Section 3]. Similarly, let $\widetilde{\mathfrak{a}} = \mathfrak{rad}_{\widetilde{\mathfrak{u}}}\widetilde{f}$ be the radical of the bilinear form $x, y \mapsto \widetilde{f}([x,y])$, $x, y \in \widetilde{\mathfrak{u}}$. Then $\dim\widetilde{\Omega} = \operatorname{codim}_{\widetilde{\mathfrak{u}}}\widetilde{\mathfrak{a}} = |\widetilde{\Phi}^+| - \dim\widetilde{\mathfrak{a}}$. Put $\mathfrak{u}_{\mathcal{A}} = \sum_{\alpha\in\mathcal{A}} k e_\alpha$ (so $\mathfrak{u} = \mathfrak{u}_{\mathcal{A}} \oplus \widetilde{\mathfrak{u}}$ as vector spaces) and $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{u}_{\mathcal{A}}$.

Lemma 3.2. *The subalgebra $\mathfrak{a}$ coincides with the direct sum of its subspaces $\mathfrak{b}$ and $\widetilde{\mathfrak{a}}$, i.e., $\mathfrak{a} = \mathfrak{b} \oplus \widetilde{\mathfrak{a}}$.*

Proof. i) For any $x = \sum_{\alpha\in\Phi^+} b_\alpha e_\alpha \in \mathfrak{u}$ let $\operatorname{Supp}(x) = \{\alpha \in \Phi^+ \mid b_\alpha \neq 0\}$. Let $\mathfrak{r} = \mathfrak{rad}_{\mathfrak{u}}\xi_\beta e_\beta^* = \mathfrak{rad}_{\mathfrak{u}} e_\beta^* = \langle e_\alpha \mid \alpha \in \Phi^+ \setminus S(\beta)\rangle_k \supset \widetilde{\mathfrak{u}}$. Suppose that there exists $x \in \widetilde{\mathfrak{a}}$ such that $x \notin \mathfrak{a} \cap \widetilde{\mathfrak{u}}$. Then there exists $y \in \mathfrak{u}$ such that $f([x,y]) \neq 0$. Since $x \in \widetilde{\mathfrak{u}} \subset \mathfrak{r}$, we see that $\xi_\beta e_\beta^*([x,y]) = 0$, so $\widetilde{f}([x,y]) \neq 0$. Hence there exist $\alpha \in \operatorname{Supp}(x)$, $\gamma \in \mathcal{A} \cap \operatorname{Supp}(y)$ such that $\alpha + \gamma = \widetilde{\beta} \in \widetilde{D}$. Using the orthogonality of the subset D and the fact that $\gamma \in \mathcal{A}$, we get $(\alpha, \beta) = (\widetilde{\beta} - \gamma, \beta) = (\widetilde{\beta}, \beta) - (\gamma, \beta) = -(\gamma, \beta) \neq 0$. This stands in contradiction with the choice of $\alpha \in \widetilde{\Phi}^+$. Thus, $x \in \mathfrak{a} \cap \widetilde{\mathfrak{u}}$, so $\widetilde{\mathfrak{a}} \subset \mathfrak{a} \cap \widetilde{\mathfrak{u}} \subset \mathfrak{a}$.

ii) On the other hand, let $x = y + z \in \mathfrak{a}$, where $y \in \mathfrak{u}_{\mathcal{A}}$, $z \in \widetilde{\mathfrak{u}}$. If $\gamma \in \widetilde{\Phi}^+$, then $\gamma \notin S(\beta)$ and $\alpha + \gamma \in \mathcal{A}$ for all $\alpha \in \mathcal{A}$. Hence $f([x, e_\gamma]) = \xi_\beta e_\beta^*([x, e_\gamma]) + \widetilde{f}([y, e_\gamma]) + \widetilde{f}([z, e_\gamma]) = \widetilde{f}([z, e_\gamma]) = 0$, i.e., $z \in \widetilde{\mathfrak{a}}$. According to the step i), $z \in \mathfrak{a} \cap \widetilde{\mathfrak{u}} \subset \mathfrak{a}$. Consequently $y = x - z \in \mathfrak{a}$, so $y \in \mathfrak{a} \cap \mathfrak{u}_{\mathcal{A}} = \mathfrak{b}$ and $\mathfrak{a} = \mathfrak{b} + \widetilde{\mathfrak{a}}$. But $\mathfrak{b} \cap \widetilde{\mathfrak{a}} = 0$, so the sum is direct. This concludes the proof. $\square$

3.3. To prove the inequality $\dim\Omega \leqslant l(\sigma) - s(\sigma)$, we need the following key observation.

Lemma 3.3. *The inequality $\#\{\alpha \in \mathcal{A} \mid \sigma\alpha > 0\} + 1 \leqslant \dim\mathfrak{b}$ holds.*

Proof. Let $\widetilde{\mathcal{A}} = \{\alpha \in \mathcal{A} \mid \sigma\alpha > 0\} \cup \{\beta\}$ (clearly, $\sigma\beta = -\beta < 0$). It's enough to construct a linearly independent set $\{x_\alpha\}_{\alpha\in\widetilde{\mathcal{A}}} \subset \mathfrak{b}$. Since $(\beta, \beta) = 1$ and β is not singular to any root from D, $\beta \in \mathcal{A}$ and $x_\beta = e_\beta \in \mathfrak{b}$.

It's convenient to split the set $\widetilde{\mathcal{A}}$ into a union $\widetilde{\mathcal{A}} = \mathcal{A}^+ \cup \mathcal{A}^- \cup \{\beta\}$, where

$$\mathcal{A}^+ = \{\alpha \in \mathcal{A} \mid \sigma\alpha > 0 \text{ and } (\alpha, \beta) > 0\},$$
$$\mathcal{A}^- = \{\alpha \in \mathcal{A} \mid \sigma\alpha > 0 \text{ and } (\alpha, \beta) < 0\}.$$

Let's consider two different cases, $\alpha \in \mathcal{A}^-$ and $\alpha \in \mathcal{A}^+$.

i) First, let $\alpha \in \mathcal{A}^-$, i.e., $(\alpha, \beta) = -1/2 < 0$ and $\sigma\alpha > 0$. Suppose α is singular to the roots $\beta_1, \ldots, \beta_l \in D$ and is not singular to any other root from D. Put $\gamma_i = \beta_i - \alpha \in \Phi^+$ for all $i = 1, \ldots, l$. Then

$$(\beta, \gamma_i) = (\beta, \beta_i - \alpha) = -(\beta, \alpha) = 1/2,$$

so either $\gamma_i \in S(\beta)$ or $\beta \in S(\gamma_i)$. But if the second case occurs, then $\beta < \beta_i$, a contradiction with the choice of the root β. Thus, $\gamma_i \in S(\beta)$ for all i, Put $\alpha_i = \beta - \gamma_i \in \Phi^+$, $i = 1, \ldots, l$.

Now set $b_i = -(\xi_{\beta_i} \cdot N_{\alpha\gamma_i})/(\xi_\beta \cdot N_{\alpha_i\gamma_i})$, $i = 1, \ldots, l$, and $x_\alpha = e_\alpha + \sum_{i=1}^l b_i e_{\alpha_i}$. Clearly, $x_\alpha \in \mathfrak{u}_{\mathcal{A}}$. We claim that $x_\alpha \in \mathfrak{a} = \mathfrak{rad}_{\mathfrak{u}} f$. Indeed, let δ be a positive root. By definition,

$$f([x_\alpha, e_\delta]) = N_{\alpha\delta} \cdot f(e_{\alpha+\delta}) + \sum_{i=1}^l N_{\alpha_i\delta} \cdot f(e_{\alpha_i+\delta}) \cdot b_i.$$

Pick a number i. We note that $\alpha_i+\delta\notin D$ if $\delta\neq\gamma_i$. Indeed, assume the converse. Then there exists $\widetilde{\beta}\in D$ such that $\widetilde{\beta}\neq\beta$ and $\alpha_i+\delta=\widetilde{\beta}$. (Obviously, $\delta\neq\gamma_i$ is equivalent to $\widetilde{\beta}\neq\beta$.) Since $(\widetilde{\beta},\alpha_i)=1/2$, $(\widetilde{\beta},\gamma_i)=(\widetilde{\beta},\beta-\alpha_i)=-1/2$. Hence $(\widetilde{\beta},\alpha)=(\widetilde{\beta},\beta_i-\gamma_i)=1/2$. Let $\beta_1',\ldots,\beta_s'$ be all the roots from D, which aren't orthogonal to α except the roots $\beta,\beta_i,\widetilde{\beta}$. Then

$$\sigma\alpha=\alpha+\beta-\beta_i+\widetilde{\beta}-2(\alpha,\beta_1')\cdot\beta_1'-\ldots-2(\alpha,\beta_s')\cdot\beta_s'.$$

Since $(\alpha,\beta_r')=\pm1/2$ for all $1\leqslant r\leqslant s$, we obtain

$$||\sigma\alpha-\alpha||^2=2-2(\sigma\alpha,\alpha)=||\beta||^2+||\beta_i||^2+||\widetilde{\beta}||^2+\sum_{r=1}^{s}||\beta_r'||^2=3+s.$$

We see that either $s=0$ or $s=1$, because $(\sigma\alpha,\alpha)\geqslant-1$. If $s=0$, then

$$\sigma\alpha=\alpha+\beta-\beta_i-\widetilde{\beta}=\alpha+(\alpha_i+\gamma_i)-(\alpha+\gamma_i)-(\alpha_i+\delta)=-\delta<0.$$

On the other hand, $(\alpha,\beta_i)=1/2$, because $\alpha\in S(\beta_i)$, so if $s=1$, then

$$\begin{aligned}(\sigma\alpha,\alpha)&=||\alpha||^2+(\beta,\alpha)-(\beta_i,\alpha)-(\widetilde{\beta},\alpha)-2(\beta_1',\alpha)^2\\&=1-1/2-1/2-1/2-1/2=-1,\end{aligned}$$

i.e., $\sigma\alpha=-\alpha<0$. By the way, $\sigma\alpha<0$. This contradicts the choice of α. We conclude that $\alpha_i+\delta\notin D$ if $\delta\neq\gamma_i$.

Hence if $\delta=\gamma_i$ for some i, then

$$\begin{aligned}f([x_\alpha,e_\delta])&=f([x_\alpha,e_{\gamma_i}])=N_{\alpha\gamma_i}\cdot\xi_{\beta_i}+N_{\alpha_i\gamma_i}\cdot\xi_\beta\cdot b_i\\&=N_{\alpha\gamma_i}\cdot\xi_{\beta_i}-N_{\alpha_i\gamma_i}\cdot\xi_\beta\cdot(\xi_{\beta_i}\cdot N_{\alpha\gamma_i})/(\xi_\beta\cdot N_{\alpha_i\gamma_i})\\&=N_{\alpha\gamma_i}\cdot\xi_{\beta_i}-N_{\alpha\gamma_i}\cdot\xi_{\beta_i}=0.\end{aligned}$$

On the other hand, if $\delta\neq\gamma_i$ for all i, then $\alpha_i+\delta\notin D$, $1\leqslant i\leqslant l$, as above. But $\alpha+\delta\notin D$, so $f([x_\alpha,e_\delta])=0$ in the case. Whence for a given $\alpha\in\mathcal{A}^-$ the vector x_α belongs to $\mathfrak{b}=\mathfrak{a}\cap\mathfrak{u}_{\mathcal{A}}$ as required.

ii) Let us now consider the case $\alpha\in\mathcal{A}^+$, i.e., $(\alpha,\beta)=1/2>0$ and $\sigma\alpha>0$. The Weyl group acts by orthogonal transformations, so $(\beta,\sigma\alpha)=(\sigma\beta,\alpha)=(-\beta,\alpha)=-1/2$. This yields that $\beta+\sigma\alpha\in\Phi^+$. If $\beta+\sigma\alpha\in S(\widetilde{\beta})$ for a some root $\widetilde{\beta}\in D$, then $\beta<\widetilde{\beta}$. This contradicts the choice of β. Thus, for a given root $\alpha\in\mathcal{A}^+$, the vector $x_\alpha=e_{\beta+\sigma\alpha}$ belongs to $\mathfrak{b}$. Note also that $(\beta,\beta+\sigma\alpha)=1-1/2=1/2\neq0$, so $\beta+\sigma\alpha\in\mathcal{A}$.

For a given root $\alpha\in\widetilde{\mathcal{A}}$, we constructed the vector $x_\alpha\in\mathfrak{b}$. It remains to check that the vectors x_α, $\alpha\in\mathcal{A}$, are linearly independent. Since $\beta+\sigma\alpha$, $\alpha\in\mathcal{A}^+$, are distinct, the corresponding vectors $x_\alpha=e_{\beta+\sigma\alpha}$ are linearly independent. If $\alpha\in\mathcal{A}^-$, then $e_\alpha\in\mathrm{Supp}(x_\alpha)$ and $\mathrm{Supp}(x_\alpha)\setminus\{\alpha\}\subset S(\beta)$. Consequently x_α, $\alpha\in\mathcal{A}^-$, are linearly independent, too. Their union with x_α, $\alpha\in\mathcal{A}^+$, is also linearly independent, because $(\mathcal{A}^-\cup S(\beta))\cap(\beta+\sigma\mathcal{A}^+)=\varnothing$. Indeed, the inner products of β with roots from $\mathcal{A}^-$ (resp. from $\beta+\sigma\mathcal{A}^+$) are negative (resp. positive), so these subsets are disjoint. Finally, for a given $\alpha\in\mathcal{A}^+$, the root $\beta+\sigma\alpha\in\Phi^+$ isn't β-singular, because $\beta+\sigma\alpha>\beta$. Thus, the set $\{x_\alpha\}_{\alpha\in\widetilde{\mathcal{A}}}$ is linearly independent. This completes the proof. □

3.4. Now we'll conclude the proof of the inequality $\dim\Omega\leqslant l(\sigma)-s(\sigma)$.

Proposition 3.4. *Let Φ be a reduced irreducible simply laced root system, $D\subset\Phi^+$ an orthogonal subset, Ω an associated orbit of the group U, and $\sigma\in W$ the involution corresponding to D. Then $\dim\Omega\leqslant l(\sigma)-s(\sigma)$.*

Proof. By the above (see Subsection 3.2) and the inductive hypothesis,

$$\begin{aligned}\dim\Omega &= |\Phi^+| - \dim\mathfrak{a} = |\widetilde{\Phi}^+| + |\mathcal{A}| - \dim\mathfrak{a} - \dim\widetilde{\mathfrak{a}} + \dim\widetilde{\mathfrak{a}}\\ &= (|\widetilde{\Phi}^+| - \dim\widetilde{\mathfrak{a}}) + |\mathcal{A}| - (\dim\mathfrak{a} - \dim\widetilde{\mathfrak{a}})\\ &= \dim\widetilde{\Omega} + |\mathcal{A}| - (\dim\mathfrak{a} - \dim\widetilde{\mathfrak{a}})\\ &\leqslant l(\widetilde{\sigma}) - s(\widetilde{\sigma}) + |\mathcal{A}| - (\dim\mathfrak{a} - \dim\widetilde{\mathfrak{a}}) = l(\widetilde{\sigma}) - s(\widetilde{\sigma}) + |\mathcal{A}| - \dim\mathfrak{b}.\end{aligned}$$

It remains to check that $l(\sigma) - s(\sigma) \geqslant l(\widetilde{\sigma}) - s(\widetilde{\sigma}) + |\mathcal{A}| - \dim\mathfrak{b}$. But $s(\sigma) = s(\widetilde{\sigma}) + 1$. We note also that the reflection r_β acts on $\widetilde{\Phi}^+$ trivially, so $|\Phi_\sigma \cap \widetilde{\Phi}^+| = |\widetilde{\Phi}_{\widetilde{\sigma}}| = l(\widetilde{\sigma})$ and

$$l(\sigma) = |\Phi_\sigma| = |\Phi_\sigma \cap \widetilde{\Phi}^+| + |\Phi_\sigma \cap \mathcal{A}| = l(\widetilde{\sigma}) + \#\{\alpha \in \mathcal{A} \mid \sigma\alpha < 0\}.$$

Hence it's enough to prove that

$$|\mathcal{A}| - \#\{\alpha \in \mathcal{A} \mid \sigma\alpha < 0\} + 1 = \#\{\alpha \in \mathcal{A} \mid \sigma\alpha > 0\} + 1 \leqslant \dim\mathfrak{b},$$

but this follows immediately from Lemma 3.3. $\square$

3.5. In the remainder of the Section, we prove that $\dim\Omega$ doesn't depend on ξ. Let $\xi' = (\xi'_\beta)_{\beta\in D}$ be a set of non-zero scalars and f the canonical form on the orbit $\Omega' = \Omega_{D,\xi'}$. Put also $\mathfrak{a}' = \mathfrak{rad}_{\mathfrak{u}} f'$. Arguing as in the proof of Proposition 3.2, we conclude that $\mathfrak{a}' = \mathfrak{b}' \oplus \widetilde{\mathfrak{a}}'$ as vector spaces, where $\mathfrak{b}' = \mathfrak{a}' \cap \mathfrak{u}_{\mathcal{A}}$, $\widetilde{f}' = f'|_{\widetilde{\mathfrak{u}}}$ and $\widetilde{\mathfrak{a}}' = \mathfrak{rad}_{\widetilde{\mathfrak{u}}}\widetilde{f}'$.

Proposition 3.5. *Let Φ be a reduced irreducible sumply laced root system, $D \subset \Phi^+$ an orthogonal subset, and ξ, ξ' sets of non-zero scalars. Put $\Omega = \Omega_{D,\xi}$, $\Omega' = \Omega_{D,\xi'}$. Then $\dim\Omega = \dim\Omega'$.*

Proof. As above, $\dim\Omega = \operatorname{codim}_{\mathfrak{u}}\mathfrak{a}$ and $\dim\Omega' = \operatorname{codim}_{\mathfrak{u}}\mathfrak{a}'$, so it remains to check that $\dim\mathfrak{a} = \dim\mathfrak{a}'$. We proceed by induction on the rank of Φ. The base ($\operatorname{rk}\Phi = 1$, i.e., $\Phi = A_1$) is evident. But $\mathfrak{a} = \mathfrak{b} \oplus \widetilde{\mathfrak{a}}$, $\mathfrak{a}' = \mathfrak{b}' \oplus \widetilde{\mathfrak{a}}'$, and $\dim\widetilde{\mathfrak{a}} = \dim\widetilde{\mathfrak{a}}'$ by an inductive assumption, since $\operatorname{rk}\widetilde{\Phi} < \operatorname{rk}\Phi$. Thus, it's enough to show that $\dim\mathfrak{b} = \dim\mathfrak{b}'$.

Obviously, it's enough to prove that $\dim\mathfrak{b} \leqslant \dim\mathfrak{b}'$. Set $x = \sum_{\alpha\in\mathcal{A}} x_\alpha e_\alpha \in \mathfrak{b}$. Put $y = \varphi(x) = \sum_{\alpha\in\mathrm{Supp}(x)} y_\alpha e_\alpha$. In the next Subsection we prove that there exist y_α such that $y \in \mathfrak{b}'$ and if the vectors $x_1, \ldots, x_m$ are linearly independent, then the vectors $\varphi(x_1), \ldots, \varphi(x_m)$ are linearly independent, too. Applying this to an arbitrary basis x_i of the space $\mathfrak{b}$, we'll obtain the result. $\square$

3.6. In this Subsection, we conclude the proof of Proposition 3.5. Our first goal is to determine the coefficients y_α. We set $y_\alpha = x_\alpha$ for all $\alpha \in \mathcal{A}$ except the following four cases.

i) There exists $\alpha_0, \gamma \in \Phi^+$, $\beta_0 \in D$ such that

$$\beta = \alpha + \gamma, \quad \beta_0 = \alpha_0 + \gamma,$$

and α, α_0 aren't singular to any other root from D. Then we put $y_\alpha = x_\alpha \cdot \xi_\beta / \xi'_\beta$.

ii) There exist $\widetilde{\alpha}, \alpha_0, \gamma, \widetilde{\gamma} \in \Phi^+$, $\beta_0, \widetilde{\beta}_0 \in D$ such that

$$\beta = \alpha + \gamma = \widetilde{\alpha} + \widetilde{\gamma}, \quad \beta_0 = \alpha_0 + \gamma, \quad \widetilde{\beta}_0 = \alpha_0 + \widetilde{\gamma},$$

and $\alpha, \widetilde{\alpha}, \gamma, \widetilde{\gamma}$ aren't singular to any other root from the subset D. Here we let $y_\alpha = x_\alpha \cdot (\xi_\beta \cdot \xi'_{\beta_0})/(\xi'_\beta \cdot \xi_{\beta_0})$. Since the conditions above are invariant under the interchanging α and $\widetilde{\alpha}$, we also put $y_{\widetilde{\alpha}} = x_{\widetilde{\alpha}} \cdot (\xi_\beta \cdot \xi'_{\widetilde{\beta}_0})/(\xi'_\beta \cdot \xi_{\widetilde{\beta}_0})$.

iii) There exist $\widetilde{\alpha}, \alpha_0, \gamma, \widetilde{\gamma}, \gamma_0 \in \Phi^+$, $\beta_0, \widetilde{\beta}_0 \in D$ such that

$$\begin{aligned}\beta &= \alpha + \gamma = \widetilde{\alpha} + \widetilde{\gamma},\\ \beta_0 &= \alpha_0 + \gamma = \widetilde{\alpha} + \gamma_0,\\ \widetilde{\beta}_0 &= \alpha + \gamma_0 = \alpha_0 + \widetilde{\gamma},\end{aligned}$$

and $\alpha, \widetilde{\alpha}, \alpha_0, \gamma, \widetilde{\gamma}, \gamma_0$ aren't orthogonal to any other root from D. As above, we set $y_\alpha = x_\alpha \cdot (\xi_\beta \cdot \xi'_{\beta_0})/(\xi'_\beta \cdot \xi_{\beta_0})$. Since the conditions are invariant under the interchanging α and $\widetilde{\alpha}$, we also put $y_{\widetilde{\alpha}} = x_{\widetilde{\alpha}} \cdot (\xi_\beta \cdot \xi'_{\widetilde{\beta}_0})/(\xi'_\beta \cdot \xi_{\widetilde{\beta}_0})$.

iv) There exist $\alpha', \gamma' \in \Phi^+$, $\beta_0 \in D$ such that $\alpha = \alpha_0$,

$$\beta = \alpha' + \gamma', \quad \beta_0 = \alpha_0 + \gamma',$$

and $\alpha = \alpha_0$, α' aren't singular to any other root from the subset D. Then we let $y_{\alpha_0} = x_{\alpha_0} \cdot \xi_{\beta_0}/\xi'_{\beta_0}$.

Let us check that y_α are well-defined. Suppose $\alpha \in \mathcal{A}$. If $\alpha \notin \mathrm{Supp}(x)$, then $y_\alpha = x_\alpha = 0$, so let $\alpha \in \mathrm{Supp}(x)$. If α is not singular to any root from D, then $y_\alpha = x_\alpha$. On the other hand, suppose $\alpha \in S(\beta)$ (i.e., $\beta = \alpha + \gamma$, $\gamma \in \Phi^+$). If γ is not singular to any other root from D, then $f([x, e_\gamma]) = \xi_\beta \cdot x_\alpha \cdot N_{\alpha\gamma} \neq 0$. This stands in contradiction with the choice of $x \in \mathfrak{b} \subset \mathfrak{a} = \mathfrak{rad_u} f$, so there exist $\alpha_0 \in \mathrm{Supp}(x)$, $\beta_0 \in D$ such that $\beta_0 \neq \beta$ and $\beta_0 = \alpha_0 + \gamma$. Suppose there exist $\widetilde{\beta}_0 \in D$, $\gamma_0 \in \Phi^+$ such that $\widetilde{\beta}_0 \neq \beta$ and $\widetilde{\beta}_0 = \alpha + \gamma_0$. If $\widetilde{\beta}_0 = \beta_0$, then

$$(\beta_0, \beta) = (\beta_0, \alpha + \gamma) = (\widetilde{\beta}_0, \alpha) + (\beta_0, \gamma) = 1/2 + 1/2 = 1,$$

a contradiction with the orthogonality of D. Hence $\widetilde{\beta}_0 \neq \beta_0$.

If γ_0 isn't singular to any root from the subset D except $\widetilde{\beta}_0$, then $f([x, e_{\gamma_0}]) = \xi_{\widetilde{\beta}_0} \cdot x_\alpha \cdot N_{\alpha\gamma_0} \neq 0)$, a contradiction. Whence there exist $\widehat{\beta} \neq \widetilde{\beta}_0$ such that $\widehat{\beta} = \widehat{\alpha} + \gamma_0$. If $\widehat{\beta} \neq \beta_0$, then consider the set $\Psi = \langle \beta, \widehat{\beta}, \beta_0, \widetilde{\beta}_0, \alpha \rangle_{\mathbb{Z}} \cap \Phi$. Clearly, Ψ is a root system of rank 5, and $\gamma, \gamma_0, \alpha_0, \widehat{\alpha} \in \Psi$. Further, $\Psi \cap \Phi^+$ is a *closed* subset of Ψ, i.e., if $\zeta_1, \zeta_2 \in \Psi \cap \Phi^+$ and $\zeta_1 + \zeta_2 \in \Psi$, then $\zeta_1 + \zeta_2 \in \Psi \cap \Phi^+$. According to [6, §16, Exercise 3], all roots from $\Psi \cap \Phi^+$ are positive with respect to some subsystem of fundamental roots of Ψ. Thus, without loss of generality, we can assume $\beta, \widehat{\beta}, \beta_0, \widetilde{\beta}_0, \gamma, \gamma_0, \alpha, \alpha_0, \widehat{\alpha}$ belongs to Ψ^+.

Since the sum of two roots from different irreducible components of a root system is not a root, Ψ is an irreducible simply laced root system. But there are *no* four pairwise orthogonal roots in A_5^+, so $\Psi \cong D_5$. Then the roots $\eta = \beta$, $\eta_1 = \widetilde{\beta}_0$, $\eta_2 = \beta_0$, $\eta_3 = \widehat{\beta}$, $\theta = \gamma$, $\theta_1 = \gamma_0$, $\psi = \alpha$, $\psi_2 = \alpha_0$ and $\psi_3 = \widehat{\alpha}$ form a non-admissible subset of D_5^+ of type 3. This contradicts Lemma 2.5.

We conclude that $\widehat{\beta} = \beta_0 = \alpha_0 + \gamma = \widehat{\alpha} + \gamma_0$. Denote $\widetilde{\alpha} = \widehat{\alpha}$, so $\beta_0 = \alpha_0 + \gamma = \widetilde{\alpha} + \gamma_0$. Then $\widetilde{\alpha} = \beta_0 - \gamma_0 = \beta_0 - (\widetilde{\beta}_0 - \alpha) = \beta_0 - \widetilde{\beta}_0 + \alpha$, so $(\beta, \widetilde{\alpha}) = (\beta, \alpha) = 1/2$. Hence $\widetilde{\alpha} \in S(\beta)$, because if $\beta \in S(\widetilde{\alpha})$, then $\beta < \beta_0$, and β is not maximal among all roots from D. In other words, $\beta = \widetilde{\alpha} + \widetilde{\gamma}$ for some $\widetilde{\gamma} \in \Phi^+$. We see that

$$\widetilde{\beta}_0 = \alpha + \gamma_0 = (\beta - \gamma) + (\beta_0 - \widetilde{\alpha}) = (\beta_0 - \gamma) + (\beta - \widetilde{\alpha}) = \alpha_0 + \widetilde{\gamma}.$$

Lemma 2.5 shows that $\alpha, \widetilde{\alpha}, \alpha_0, \gamma, \widetilde{\gamma}, \gamma_0$ aren't orthogonal to any other root $\beta_2 \in D$. Indeed, assume the converse, Then the roots $\eta = \beta$, $\eta_1 = \beta_0$, $\eta' = \widetilde{\beta}_0$, $\theta = \alpha$, $\theta_1 = \alpha_0$, $\theta'_1 = \widetilde{\alpha}$, $\psi = \gamma$, $\psi' = \gamma_0$, $\psi'_1 = \widetilde{\gamma}$ and $\eta_2 = \beta_2$ form a non-admissible subset of D_5^+ of type 4. Thus, α belongs to case iii), and the roots $\widetilde{\alpha}, \alpha_0, \beta_0, \widetilde{\beta}_0, \gamma, \widetilde{\gamma}, \gamma_0$ are determined uniquely.

Suppose now that $\beta = \alpha + \gamma$, $\beta_0 = \alpha_0 + \gamma$, but α is not singular to any other root from D except β. Suppose also that there exists $\widetilde{\gamma} \in \Phi^+$ such that $\widetilde{\beta}_0 = \alpha_0 + \widetilde{\gamma}$. Then

$$(\beta, \widetilde{\gamma}) = (\beta, \widetilde{\beta}_0 - \alpha_0) = (\beta, \widetilde{\beta}_0 - \beta_0 + \gamma) = 1/2,$$

because $\gamma \in S(\beta)$. This implies $\widetilde{\gamma} \in S(\beta)$, because if $\beta \in S(\gamma)$, then $\beta < \beta_0$, and β is not maximal. Let $\beta = \widetilde{\alpha} + \widetilde{\gamma}$ for some $\widetilde{\alpha} \in \Phi^+$. We note that $\widetilde{\alpha}$ is not singular to any other root from D except β. Indeed, if the converse holds, then the root $\widetilde{\alpha}$ belongs to case iii). But this yields $\alpha \in S(\widetilde{\beta}_0)$, a contradiction.

Besides, γ isn't singular to any other root from D except β and β_0. Indeed, if there exist $\beta_2 \in D$, $\alpha_2 \in \Phi^+$ such that $\beta_2 \neq \beta$, $\beta_2 \neq \beta_0$ and $\beta_2 = \alpha_2 + \gamma$, then the roots $\eta = \beta$, $\eta_1 = \beta_0$, $\eta' = \widetilde{\beta}_0$, $\eta_2 = \beta_2$, $\theta = \alpha$, $\theta' = \widetilde{\alpha}$, $\theta_1 = \alpha_0$, $\theta_2 = \alpha_2$, $\psi = \gamma$ and $\psi' = \widetilde{\gamma}$ form a non-admissible subset of D_5^+ of type 1.

This stands in contradiction with Lemma 2.5. Similarly, $\widetilde{\gamma}$ isn't singular to any other root from D except β, $\widetilde{\beta}_0$. We see that α belongs to case ii), and the roots β_0, $\widetilde{\beta}_0$ are determined uniquely.

Suppose now that $\beta = \alpha + \gamma$, $\beta_0 = \alpha_0 + \gamma$ and the roots α, α_0 aren't singular to any other root from D except β, β_0 respectively, Then α belongs to case i), and the root β_0 is determined uniquely.

It remains to consider the case when $\alpha = \alpha_0$ isn't singular to β, but there exist $\beta_0 \in D$, $\gamma' \in \Phi^+$ such that $\beta_0 \neq \beta$ and $\beta_0 = \alpha_0 + \gamma'$. Since $\alpha \notin S(\beta)$, α doesn't belong to cases i)–iii). If $\gamma' \notin S(\beta)$, then $\alpha = \alpha_0$ doesn't belong to case iv), too, so $y_{\alpha_0} = x_{\alpha_0}$.

Suppose now $\gamma' \in S(\beta)$, i.e., there exists $\alpha' \in \Phi^+$ such that $\beta = \alpha' + \gamma'$. If $\alpha' \in \mathrm{Supp}(x)$, then the root α' belongs to one of cases i)–iii). This implies $y_{\alpha_0} = x_{\alpha_0}$. If $\alpha' \notin \mathrm{Supp}(x)$, then there exist $\beta_1 \in D$, $\alpha_1 \in \Phi^+$ such that $\beta \neq \beta$, $\beta_1 \neq \beta_0$ and $\beta_1 = \alpha_1 + \gamma'$ (if the converse holds, then $f([x, e_{\gamma'}]) = \xi_{\beta_0} \cdot x_{\alpha_0} \cdot N_{\alpha_0\gamma'} \neq 0$, a contradiction). Hence α', $\alpha = \alpha_0$ aren't singular to any other root from D except β, β_0 respectively.

Indeed, suppose there exist $\widetilde{\beta}_0 \in D$, $\widetilde{\gamma} \in \Phi^+$ such that $\widetilde{\beta}_0 \neq \beta_0$ and $\widetilde{\beta}_0 = \alpha_0 + \widetilde{\gamma}$. The root $\widetilde{\beta}_0$ doesn't coincide with β, because $\alpha = \alpha_0$ isn't singular to β. If $\widetilde{\beta}_0$ coincides with β_1, then

$$(\beta_0, \beta_1) = (\alpha_0, \gamma', \beta_1) = (\alpha_0, \widetilde{\beta}_0) + (\gamma', \beta_1) = 1/2 + 1/2 = 1.$$

At the same time $\widetilde{\gamma} = \widetilde{\beta}_0 - \alpha_0 = \widetilde{\beta}_0 - \beta_0 + \gamma'$, so $(\beta, \widetilde{\gamma}) = 1/2$ and $\widetilde{\gamma} \in S(\beta)$ (if $\beta \in S(\widetilde{\gamma})$, then $\beta < \widetilde{\beta}_0$, so β isn't maximal). However if $\beta = \widetilde{\alpha} + \widetilde{\gamma}$, $\widetilde{\alpha} \in \Phi^+$, then the roots $\eta = \beta$, $\eta_1 = \beta_0$, $\eta_2 = \beta_1$, $\eta' = \widetilde{\beta}_0$, $\theta = \alpha'$, $\theta_1 = \alpha = \alpha_0$, $\theta_2 = \alpha_1$, $\psi = \gamma'$ and $\psi' = \widetilde{\gamma}$ form a non-admissible subset of D_5^+ of type 1, a contradiction with Lemma 2.5. This contradiction shows that $\widetilde{\beta}_0 \neq \beta_1$.

On the other hand, suppose that there exist $\widetilde{\beta} \in D$, $\widetilde{\gamma} \in \Phi^+$ such that $\widetilde{\beta} \neq \beta$ and $\widetilde{\beta} = \alpha' + \widetilde{\gamma}$. If $\widetilde{\beta}$ coincides with β_0, then $(\beta, \beta_0) = 1$, so $\widetilde{\beta} \neq \beta_0$; for the same reason, $\widetilde{\beta} \neq \beta_1$. It follows that the roots $\eta = \beta$, $\eta' = \widetilde{\beta}$, $\eta_1 = \beta_0$, $\eta_2 = \beta_1$, $\theta = \gamma$, $\theta' = \widetilde{\gamma}$, $\psi = \alpha'$, $\psi_1 = \alpha = \alpha_0$ and $\psi_2 = \alpha_1$ form a non-admissible subset of D_5^+ of type 2. This contradicts Lemma 2.5.

We have proved that if $\alpha = \alpha_0 \notin S(\beta)$, $\alpha \in S(\beta_0)$ for some $\beta_0 \in D$, $\beta_0 \neq \beta$, and $y_\alpha \neq x_\alpha$, then $\alpha = \alpha_0$ belongs to case iv); in particular β_0 is determined uniquely.

Therefore if $y_\alpha \neq x_\alpha$, then α belongs to one of cases i)–iv), and the root β_0 is determined uniquely. Since y_α depend only on β_0, they are well-defined. Denote by X, Y the $(|\mathcal{A}| \times l)$-matrices whose columns consist of the coordinates of the vectors $x_1, \ldots, x_l \in \mathfrak{b}$ and $y_1 = \varphi(x_1), \ldots,\ y_l = \varphi(x_l)$ respectively in the basis $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ of the algebra $\mathfrak{u}_{\mathcal{A}}$. Let T be the diagonal $(|\mathcal{A}| \times |\mathcal{A}|)$-matrix whose (α, α)-th element equals

$$t_{\alpha,\alpha} = \begin{cases} \xi_\beta/\xi'_\beta, & \text{if } \alpha \text{ belongs to case i)}, \\ (\xi_\beta \cdot \xi'_{\beta_0})/(\xi'_\beta \cdot \xi_{\beta_0}), & \text{if } \alpha \text{ belongs either to cases ii) or iii)}, \\ \xi_{\beta_0}/\xi'_{\beta_0}, & \text{if } \alpha = \alpha_0 \text{ belongs to case iv)}, \\ 1 & \text{otherwise.} \end{cases}$$

We see that $Y = TX$, but $\det T \neq 0$, so $\mathrm{rk}\, X = \mathrm{rk}\, Y$. To conclude the proof, it remains to check that if $x \in \mathfrak{b}$, then $y \in \mathfrak{b}'$, i.e., $f'([y, e_{\widehat{\gamma}}]) = 0$ for all $\widehat{\gamma} \in \Phi^+$. Let us consider four cases.

1. Firstly, suppose $(\beta, \widehat{\gamma}) = 0$, i.e., $\widehat{\gamma} \in \widetilde{\Phi}^+$ and $\widehat{\gamma} \notin S(\beta)$. Let $\widehat{\gamma}$ be singular to the roots $\beta_1, \ldots, \beta_l$ from $\widetilde{D} = D \setminus \{\beta\}$ and not singular to any other root from $\widetilde{D}$. Denote $\alpha_i = \beta_i - \widehat{\gamma}$. Then $(\beta, \alpha_i) = (\beta, \beta_i - \widehat{\gamma}) = 0$, so e_{α_i} doesn't belong to $\mathrm{Supp}(x) \subset \mathcal{A} = \Phi^+ \setminus \widetilde{\Phi}^+$. Thus, $f'([y, e_{\widehat{\gamma}}]) = 0$.

2. Secondly, suppose $(\beta, \widehat{\gamma}) \neq 0$, i.e., $\widehat{\gamma} \in \mathcal{A}$. If $(\beta, \widehat{\gamma}) = 1$, then $\widehat{\gamma} = \beta$. But β isn't singular to any root from D, so $f'([y, e_{\widehat{\gamma}}]) = f'([y, e_\beta]) = 0$. If $(\beta, \widehat{\gamma}) = -1/2$, then $\widehat{\gamma}$ isn't singular to β; in this case, denote $\gamma_0 = \widehat{\gamma}$ (so we must prove that $f'([y, e_{\gamma_0}]) = 0$). If $\alpha = \widetilde{\beta}_0 - \gamma_0 \notin \mathrm{Supp}(x)$ for all $\widetilde{\beta}_0 \in \widetilde{D}$ such that $\gamma_0 \in S(\beta_0)$, then $f'([y, e_{\gamma_0}]) = 0$. On the other hand, suppose there exist $\widetilde{\beta}_0 \in \widetilde{D}$, $\alpha \in \mathrm{Supp}(x)$ such that $\widetilde{\beta}_0 = \alpha + \gamma_0$.

Since $(\beta, \alpha) = (\beta, \widetilde{\beta}_0 - \gamma) = 1/2$, $\alpha \in S(\beta)$ (if $\beta \in S(\alpha)$, then $\beta < \widetilde{\beta}_0$, so β isn't maximal). In other words, there exists $\gamma \in \Phi^+$ such that $\beta = \alpha + \gamma$. If γ isn't singular to any other root from D,

then $f([x, e_\gamma]) = \xi_\beta \cdot x_\alpha \cdot N_{\alpha\gamma} \neq 0$, a contradiction. Whence there exist $\beta_0 \in D$, $\beta_0 \neq \beta$, $\alpha_0 \in \Phi^+$ such that $\beta_0 = \alpha_0 + \gamma$. Arguing as above, we see that α belongs to case iii), and, consequently,

$$\begin{aligned}
f'([y, e_{\gamma_0}]) &= \xi'_{\beta_0} \cdot N_{\widetilde{\alpha}\gamma_0} \cdot y_{\widetilde{\alpha}} + \xi'_{\widetilde{\beta}_0} \cdot N_{\alpha\gamma_0} \cdot y_\alpha \\
&= \xi'_{\beta_0} \cdot N_{\widetilde{\alpha}\gamma_0} \cdot x_{\widetilde{\alpha}} \cdot (\xi_\beta \cdot \xi'_{\widetilde{\beta}_0})/(\xi'_\beta \cdot \xi_{\widetilde{\beta}_0}) \\
&+ \xi'_{\widetilde{\beta}_0} \cdot N_{\alpha\gamma_0} \cdot y_\alpha \cdot (\xi_\beta \cdot \xi'_{\beta_0})/(\xi'_\beta \cdot \xi_{\beta_0}) \\
&= (\xi_{\beta_0} \cdot N_{\widetilde{\alpha}\gamma_0} \cdot x_{\widetilde{\alpha}} + \xi_{\widetilde{\beta}_0} \cdot N_{\alpha\gamma_0} \cdot x_\alpha) \cdot (\xi'_{\beta_0} \cdot \xi'_{\widetilde{\beta}_0} \cdot \xi_\beta)/(\xi_{\beta_0} \cdot \xi_{\widetilde{\beta}_0} \cdot \xi'_\beta) \\
&= f([x, e_{\gamma_0}]) \cdot (\xi'_{\beta_0} \cdot \xi'_{\widetilde{\beta}_0} \cdot \xi_\beta)/(\xi_{\beta_0} \cdot \xi_{\widetilde{\beta}_0} \cdot \xi'_\beta) = 0.
\end{aligned}$$

3. Thirdly, suppose $(\beta, \widehat{\gamma}) = 1/2$ and $\widehat{\gamma} \in S(\beta)$; in this case, denote $\gamma = \widehat{\gamma}$ (so we must prove that $f'([y, e_\gamma]) = 0$). Let $\beta = \alpha + \gamma$, $\alpha \in \Phi^+$. Let also γ be singular to the roots $\beta_1, \dots, \beta_l$ from $\widetilde{D}$ and not singular to any other root from $\widetilde{D}$. Denote $\alpha_i = \beta_i - \gamma$. If $\alpha \in \mathrm{Supp}(x)$, then, arguing as above, we conclude that α belongs to one of cases i)–iii). If α belongs to case i), then all the roots α_i belong to case iv). Hence

$$\begin{aligned}
f'([y, e_\gamma]) &= \xi'_\beta \cdot N_{\alpha\gamma} \cdot y_\alpha + \sum\nolimits_{i=1}^{l} \xi'_{\beta_i} \cdot N_{\alpha_i\gamma} \cdot y_{\alpha_i} \\
&= \xi'_\beta \cdot N_{\alpha\gamma} \cdot x_\alpha \cdot \xi_\beta/\xi'_\beta + \sum\nolimits_{i=1}^{l} \xi'_{\beta_i} \cdot N_{\alpha_i\gamma} \cdot x_{\alpha_i} \cdot \xi_{\beta_i}/\xi'_{\beta_i} \\
&= \xi_\beta \cdot N_{\alpha\gamma} \cdot x_\alpha + \sum\nolimits_{i=1}^{l} \xi_{\beta_i} \cdot N_{\alpha_i\gamma} \cdot x_{\alpha_i} = f([x, e_\gamma]) = 0.
\end{aligned}$$

On the other hand, if α belongs either to case ii) or iii), then γ is not singular to any other root from D except β and $\beta_0 = \alpha_0 + \gamma$, $y_{\alpha_0} = x_{\alpha_0}$, and

$$\begin{aligned}
f'([y, e_\gamma]) &= \xi'_\beta \cdot N_{\alpha\gamma} \cdot y_\alpha + \xi'_{\beta_0} \cdot N_{\alpha_0\gamma} \cdot y_{\alpha_0} \\
&= \xi'_\beta \cdot N_{\alpha\gamma} \cdot x_\alpha \cdot (\xi_\beta \cdot \xi'_{\beta_0})/(\xi_{\beta'} \cdot \xi_{\beta_0}) + \xi'_{\beta_0} \cdot N_{\alpha_0\gamma} \cdot x_{\alpha_0} \\
&= (\xi_\beta \cdot N_{\alpha\gamma} \cdot x_\alpha + \xi_{\beta_0} \cdot N_{\alpha_0\gamma} \cdot x_{\alpha_0}) \cdot \xi'_{\beta_0}/\xi_{\beta_0} \\
&= f([x, e_\gamma]) \cdot \xi'_{\beta_0}/\xi_{\beta_0} = 0.
\end{aligned}$$

Assume now that $\alpha \notin \mathrm{Supp}(x)$. If $\alpha_i \notin \mathrm{Supp}(x)$ for all i, then $f'([y, e_\gamma]) = 0$, because $\mathrm{Supp}(x) = \mathrm{Supp}(y)$. At the same time if $\alpha_i \in \mathrm{Supp}(x)$, then there exists α_j such that $i \neq j$ and $\alpha_j \in \mathrm{Supp}(x)$. We claim that the root α_i isn't singular to any other root from D except β_i. Indeed, assume $\widetilde{\beta}_i = \alpha_i + \widetilde{\gamma}$ for some $\widetilde{\beta}_i \in D$, $\widetilde{\gamma} \in \Phi^+$. Since $(\beta, \alpha_i) = (\beta, \beta_i - \gamma) = -1/2$, $\alpha_i \notin S(\beta)$, so $\widetilde{\beta}_i \neq \beta$. It follows that $(\beta, \widetilde{\gamma}) = (\beta, \widetilde{\beta}_i - \alpha_i) = 1/2$, so $\widetilde{\gamma} \in S(\beta)$ (if $\beta \in S(\widetilde{\gamma})$, then $\beta < \widetilde{\beta}_i$, so β isn't maximal). Put $\beta = \widetilde{\alpha} + \widetilde{\gamma}$, $\widetilde{\alpha} \in \Phi^+$. If $\widetilde{\beta}_i = \beta_j$, then

$$(\beta_i, \beta_j) = (\alpha_i + \gamma, \beta_j) = (\alpha_i, \widetilde{\beta}_i) + (\gamma, \beta_j) = 1/2 + 1/2 = 1.$$

Hence $\widetilde{\beta}_i \neq \beta_j$, so the roots $\eta = \beta$, $\eta_1 = \beta_i$, $\eta_2 = \beta_j$, $\eta' = \widetilde{\beta}_i$, $\theta = \alpha$, $\theta' = \widetilde{\alpha}$, $\theta_1 = \alpha_i$, $\theta_2 = \alpha_j$, $\psi = \gamma$ and $\psi' = \widetilde{\gamma}$ form a non-admissible subset of D_5^+ of type 1, a contradiction with Lemma 2.5. This contradiction shows that α_i isn't singular to any other root from D except β_i, as required.

Similarly, suppose there exist $\widetilde{\beta} \in D$, $\widetilde{\gamma} \in \Phi^+$ such that $\widetilde{\beta} \neq \beta$ and $\widetilde{\beta} = \alpha + \widetilde{\gamma}$. If $\widetilde{\beta}$ coincides with β_i, then

$$(\beta, \beta_i) = (\alpha + \gamma, \beta_i) = (\alpha, \widetilde{\beta}) + (\gamma, \beta_i) = 1/2 + 1/2 = 1.$$

Hence $\widetilde{\beta} \neq \beta_i$. For the same reason, $\widetilde{\beta} \neq \beta_j$. But this yields that the roots $\eta = \beta$, $\eta' = \widetilde{\beta}$, $\eta_1 = \beta_i$, $\eta_2 = \beta_j$, $\theta = \gamma$, $\theta' = \widetilde{\gamma}$, $\psi = \alpha$, $\psi_1 = \alpha_i$ and $\psi_2 = \alpha_j$ form a non-admissible subset of D_5^+ of type 2. This contradicts Lemma 2.5. We've proved that α and all $\alpha_i \in \mathrm{Supp}(x)$ aren't singular to any other root from D except β, β_i respectively. This implies that all $\alpha_i \in \mathrm{Supp}(x)$ belong to case iv), so

$$f'([y,e_\gamma])=\sum\nolimits_{i=1}^{l}\xi'_{\beta_i}\cdot N_{\alpha_i\gamma}\cdot y_{\alpha_i}=\sum\nolimits_{i=1}^{l}\xi'_{\beta_i}\cdot N_{\alpha_i\gamma}\cdot x_{\alpha_i}\cdot\xi_{\beta_i}/\xi'_{\beta_i}$$
$$=\sum\nolimits_{i=1}^{l}\xi_{\beta_i}\cdot N_{\alpha_i\gamma}\cdot x_{\alpha_i}=f([x,e_\gamma])=0.$$

4. Finally, suppose $(\beta,\widehat{\gamma})=1/2$ and $\beta\in S(\widehat{\gamma})$. Then $\widehat{\gamma}$ isn't singular to any root from D, so $f'([y,e_{\widehat{\gamma}}])=0$. This concludes the proof of Proposition 3.5.

Remark 3.6. The case $\Phi=A_n$ was considered by A.N. Panov in the paper [16]. The case $\Phi=D_n$ was considered by the author in the paper [10]. Actually, the new result is obtained only for the root systems of types E_6,E_7,E_8. However note that the *proofs* are similar for all simply laced root systems.

4. Multiply laced root systems

4.1. Throughout the section, we assume Φ to be reduced irreducible *multiply laced* root system (i.e., containing long and short roots). The cases of B_n and C_n were considered by the author in the paper [10], so we'll assume that Φ is of type F_4 or G_2. Firstly, suppose that $\Phi=G_2$ (this case is quite easy).

Recall that $G_2^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2,2\alpha_1+\alpha_2,3\alpha_1+2\alpha_2\}$, where $||\alpha_1||^2=1$, $||\alpha_2||^2=3$ and the angle between the vectors α_1,α_2 equals $5\pi/6$. Let D be an orthogonal subset of Φ^+. Of course, $|D|\leq 2$. For $|D|=1$, there is nothing to prove, because Ω is an elementary orbit (see Example 2.7). There are three orthogonal subsets of G_2^+ of cardinality two; we'll consider all of them subsequently. Note that $l(\sigma)=6$ and $l(\sigma)-s(\sigma)=6-2=4$, because σ is the central symmetry. Denote $D=\{\beta_1,\beta_2\}$.

i) $\beta_1=\alpha_1,\beta_2=3\alpha_1+2\alpha_2$. The root α_1 is fundamental, so $S(\alpha_1)=\varnothing$. At the same time β_2-singular roots are the following: $S(\beta_2)=\{\alpha_1,3\alpha_1+\alpha_2\}\cup\{\alpha_1+\alpha_2,2\alpha_1+\alpha_2\}$. Put $\mathcal{M}=\{\alpha_2,\alpha_1+\alpha_2\}$, $\mathcal{P}=\Phi^+\setminus\mathcal{M}$ and $\mathfrak{p}=\sum_{\alpha\in\mathcal{P}}ke_\alpha$. One can see that $\mathfrak{p}\subset\mathfrak{u}$ is an *isotropic* subspace, i.e., $f([x,y])=0$ for all $x,y\in\mathfrak{p}$. (Indeed, if $1\leq i\leq 2$ and α,γ are β_i-singular, then $\mathcal{P}$ doesn't contain both of them.) Further, if $x\notin\mathfrak{p}$, then $\mathrm{Supp}(x)$ contains at least one of the roots $\gamma_1=\alpha_2,\gamma_2=\alpha_1+\alpha_2$. Actually if $x=x_1e_{\gamma_1}+\ldots$, $x_1\neq 0$, then $f([x,e_{3\alpha_1+\alpha_2}])=\xi_{\beta_2}\cdot x_1\cdot N_{\gamma_1,3\alpha_1+\alpha_2}\neq 0$. Similarly, if $x=x_2e_{\gamma_2}+\ldots$, $x_2\neq 0$, then $f([x,e_{2\alpha_1+\alpha_2}])=\xi_{\beta_2}\cdot x_2\cdot N_{\gamma_2,2\alpha_1+\alpha_2}\neq 0$. Thus, $\mathfrak{p}$ is a *maximal* isotropic subspace with respect to the inclusion order. Hence $\dim\Omega$ doesn't depend on ξ and equals $2\cdot\mathrm{codim}_{\,\mathfrak{u}}\mathfrak{p}=4=l(\sigma)-s(\sigma)$ (see, f.e., [2, Section 3]).

ii) $\beta_1=\alpha_1+\alpha_2,\beta_2=3\alpha_1+\alpha_2$. Here $S(\beta_1)=\{\alpha_1,\alpha_2\}$, $S(\beta_2)=\{\alpha_1,2\alpha_1+\alpha_2\}$. Put $\mathcal{M}=\{\alpha_1\}$. Let $\mathcal{P},\mathfrak{p}$ be defined as above. Evidently, $\mathfrak{p}$ is an isotropic subspace. On the other hand, if $x=x_1e_{\alpha_1}+\ldots\notin\mathfrak{p}$, then $f([x,e_{\alpha_2}])=\xi_{\beta_1}\cdot x_1\cdot N_{\alpha_1\alpha_2}\neq 0$, so $\mathfrak{p}$ is a maximal isotropic subspace and, consequently, $\dim\Omega$ doesn't depend on ξ and equals $2\cdot\mathrm{codim}_{\,\mathfrak{u}}\mathfrak{p}=2<4=l(\sigma)-s(\sigma)$.

iii) $\beta_1=\alpha_2,\beta_2=2\alpha_1+\alpha_2$. The root α_2 is fundamental, so $S(\alpha_2)=\varnothing$. At the same time $S(\beta_2)=\{\alpha_1,\alpha_1+\alpha_2\}$. Putting $\mathcal{M}=\{\alpha_1\}$, we see that $\dim\Omega$ doesn't depend on ξ and equals $2<4=l(\sigma)-s(\sigma)$, as in the previous step.

4.2. Let us now consider the more complicated case $\Phi=F_4$. Recall that

$$F_4^+=\{\varepsilon_i,\varepsilon_i\pm\varepsilon_j,(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)/2,1\leqslant i<j\leqslant 4\}\quad\text{(signs are independent)},$$

where $\alpha_1=\varepsilon_2-\varepsilon_3$, $\alpha_2=\varepsilon_3-\varepsilon_4$, $\alpha_3=\varepsilon_4$, $\alpha_4=(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$ are fundamental roots (here $\{\varepsilon_i\}_{i=1}^4$ is the standard basis of $\mathbb{R}^4$). For convenience, put $\widetilde{\Phi}^+=\{\varepsilon_i,\varepsilon_i\pm\varepsilon_j,1\leqslant i<j\leqslant 4\}$ and $\mathcal{B}=\Phi^+\setminus\widetilde{\Phi}^+$. One has $\widetilde{\Phi}\cong B_4$ as root systems, where $\widetilde{\Phi}=\pm\widetilde{\Phi}^+$.

We begin with the case $D\subset\widetilde{\Phi}^+$. We denote by $\widetilde{W}$ the Weyl group of the root system $\widetilde{\Phi}$. We also denote by $\widetilde{\sigma}$ the involution in the $\widetilde{W}$ corresponding to the subset D. Clearly, $s(\widetilde{\sigma})=s(\sigma)=|D|$ and $l(\widetilde{\sigma})\leqslant l(\sigma)$. Precisely, $\mathcal{F}=\widetilde{\mathcal{F}}+\#\{\alpha\in\mathcal{B}\mid\sigma\alpha<0\}$ (here we put $\mathcal{F}=l(\sigma)-s(\sigma)$ and $\widetilde{\mathcal{F}}=l(\widetilde{\sigma})-s(\widetilde{\sigma})$).

As above, denote $\widetilde{\mathfrak{u}}=\sum_{\alpha\in\widetilde{\Phi}^+}ke_\alpha$, $\mathfrak{u}_{\mathcal{B}}=\sum_{\alpha\in\mathcal{B}}ke_\alpha$ (hence $\mathfrak{u}=\widetilde{\mathfrak{u}}\oplus\mathfrak{u}_{\mathcal{B}}$ as vector spaces) and set $\widetilde{f}=f|_{\widetilde{\mathfrak{u}}}$, $\widetilde{U}=\exp(\widetilde{\mathfrak{u}})$. Let $\widetilde{\Omega}\subset\widetilde{\mathfrak{u}}^*$ be the orbit of $\widetilde{f}$ under the coadjoint action of the group $\widetilde{U}$. Let

$\mathfrak{a} = \mathfrak{rad}_{\mathfrak{u}} f$, $\widetilde{\mathfrak{a}} = \mathfrak{rad}_{\widetilde{\mathfrak{u}}} \widetilde{f}$. It follows from [10, Theorem 1.2] that $\dim \widetilde{\Omega} = \widetilde{\mathcal{F}} - \vartheta$, where ϑ depends only on D, not on ξ. Finally, put $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{u}_{\mathcal{B}}$.

Lemma 4.1. *One has* $\mathfrak{a} = \widetilde{\mathfrak{a}} \oplus \mathfrak{b}$ *as vector spaces* (*cf. Lemma* 3.2).

Proof. Suppose $x \in \widetilde{\mathfrak{a}}$, $\alpha \in \mathrm{Supp}(x)$. If $\gamma \in \widetilde{\Phi}$, then $f([x, e_\gamma]) = 0$, because $\widetilde{\mathfrak{a}} = \mathfrak{rad}_{\widetilde{\mathfrak{u}}} f$ is the radical of f. If $\gamma \in \mathcal{B}$, then $\alpha + \gamma \in \mathcal{B}$, because $f([e_\alpha, e_\gamma]) = 0$. We see that $f([x, e_\gamma]) = 0$ for all $\gamma \in \Phi^+$, hence $x \in \mathfrak{a}$. Thus, $\widetilde{\mathfrak{a}} \subset \mathfrak{a}$. On the other hand, suppose $x = y + z \in \mathfrak{a}$, $y \in \widetilde{\mathfrak{u}}$, $z \in \mathfrak{u}_{\mathcal{B}}$, $\alpha \in \mathrm{Supp}(z)$ and $\gamma \in \widetilde{\Phi}^+$. Then $\alpha + \gamma \in \mathcal{B}$, so $f([z, e_\gamma]) = 0$ and $f([y, e_\gamma]) = 0$, i.e., $y \in \widetilde{\mathfrak{a}} \subset \mathfrak{a}$. Hence $z \in \mathfrak{a}$ and $\mathfrak{a} = \widetilde{\mathfrak{a}} + \mathfrak{b}$. But $\widetilde{\mathfrak{a}} \cap \mathfrak{b} = 0$. The proof is complete. □

Lemma 4.2. *One has* $\mathfrak{b} = \langle e_\alpha, \alpha \in \mathcal{B} \mid \sigma\alpha > 0 \rangle_k$ (*cf. Lemma* 3.3).

Proof. Set $\widetilde{\mathcal{B}} = \{\alpha \in \mathcal{B} \mid \sigma\alpha > 0\}$. Firstly, suppose that D doesn't contain the roots ε_1, $\varepsilon_1 \pm \varepsilon_j$, $j = 2, 3, 4$. Then $\sigma\alpha = \varepsilon_1/2 \pm \ldots > 0$ for all $\alpha \in \mathcal{B}$, so $\widetilde{\mathcal{B}} = \mathcal{B}$. In this case, $\mathcal{B} \cap \bigcup_{\beta' \in D} S(\beta') = \varnothing$. Hence $\mathfrak{b} = \mathfrak{u}_{\mathcal{B}}$ as required.

Secondly, suppose $\beta = \varepsilon_1 \in D$. Then D doesn't contain the roots $\varepsilon_1 \pm \varepsilon_j$, $j = 2, 3, 4$, so $\sigma\alpha = -\varepsilon_1/2 \pm \ldots < 0$ for all $\alpha \in \mathcal{B}$. This implies $\widetilde{\mathcal{B}} = \varnothing$. On the other hand, if $\alpha \in \mathcal{B}$, then $\gamma = \beta - \alpha$ is not singular to any other root from D except β. Whence $f([x, e_\gamma]) = \xi_\beta \cdot x_\alpha \cdot N_{\alpha\gamma} \neq 0$ if $x = x_\alpha e_\alpha + \ldots \in \mathfrak{u}_{\mathcal{B}}$. Thus, $\mathfrak{b} = 0$ as required.

Thirdly, suppose there exists j such that $\beta = \varepsilon_1 - \varepsilon_j \in D$ and $\varepsilon_1 + \varepsilon_j \notin D$. In this case, $\sigma\alpha > 0$ if and only if $\alpha = (\varepsilon_1 + \varepsilon_j \pm \ldots)/2$. If $\gamma \in \widetilde{\Phi}^+$, then $\alpha + \gamma \in \mathcal{B}$, so $f([e_\alpha, e_\gamma]) = 0$. At the same time if $\gamma \in \mathcal{B}$, then the coefficient of ε_j in γ is not less than $-1/2$, so $\alpha + \gamma \neq \beta$. On the other hand, $\alpha + \gamma = \varepsilon_1 \pm \ldots$, so $\alpha + \gamma \notin S(\beta)$ for all $\beta \in D$. This yields that $e_\alpha \in \mathfrak{b}$. But if $x \in \mathfrak{u}_{\mathcal{B}}$ and $\alpha = (\varepsilon_1 - \varepsilon_j \pm \ldots)/2 \in \mathrm{Supp}(x)$, then $\gamma = \beta - \alpha$ isn't singular to any other root from D except β. If $x = x_\alpha e_\alpha + \ldots$, then $f([x, e_\gamma]) = \xi_\beta \cdot x_\alpha \cdot N_{\alpha\gamma} \neq 0$, a contradiction. Thus, $\mathfrak{b} = \langle e_\alpha, \alpha \in \widetilde{\mathcal{B}} \rangle_k$.

Similarly, if there exists j such that $\beta = \varepsilon_1 + \varepsilon_j \in D$ and $\varepsilon_1 - \varepsilon_j \notin D$, then $\sigma\alpha > 0$ if and only if $\alpha = (\varepsilon_1 - \varepsilon_j \pm \ldots)/2$, i.e., $e_\alpha \in \mathfrak{b}$. At the same time if $\alpha = (\varepsilon_1 + \varepsilon_j \pm \ldots)$, then $\gamma = \beta - \alpha$ isn't singular to any other root from D except β, so $f([x, e_\gamma]) \neq 0$. It follows that if $x \in \mathfrak{u}_{\mathcal{B}}$ and $\alpha \in \mathrm{Supp}(x)$, then $x \notin \mathfrak{a}$. Hence $\mathfrak{b} = \langle e_\alpha, \alpha \in \widetilde{\mathcal{B}} \rangle_k$.

Finally, suppose $\varepsilon_1 - \varepsilon_j, \varepsilon_1 + \varepsilon_j \in D$ for some j. Then $\sigma\alpha = -\varepsilon_1/2 \pm \ldots < 0$ for all $\alpha \in \mathcal{B}$, so $\widetilde{\mathcal{B}} = \varnothing$. Let α be a root from $\mathcal{B}$. Then $\alpha = (\varepsilon_1 + z \cdot \varepsilon_j \pm \ldots)/2$, $z = \pm 1$, so α isn't singular to any other root from D except $\beta = \varepsilon_1 + z \cdot \varepsilon_j$; this is also true for the root $\gamma = \beta - \alpha$. Arguing as above, we see that $x \notin \mathfrak{a}$ if $x \in \mathfrak{u}_{\mathcal{B}}$ and $\alpha \in \mathrm{Supp}(x)$. Thus, $\mathfrak{b} = 0$ as required. The proof is complete. □

It follows from two previous Lemmas that

$$\begin{aligned}
\dim \Omega = \mathrm{codim}_{\mathfrak{u}} \mathfrak{a} = \dim \mathfrak{u} - \dim \mathfrak{a} &= |\Phi^+| - (\dim \widetilde{\mathfrak{a}} + \dim \mathfrak{b}) \\
&= |\widetilde{\Phi}^+| + |\mathcal{B}| - \dim \widetilde{\mathfrak{a}} - \dim \mathfrak{b} = (|\widetilde{\Phi}^+| - \dim \widetilde{\mathfrak{a}}) + |\mathcal{B}| - \dim \mathfrak{b} \\
&= (\dim \widetilde{\mathfrak{u}} - \dim \widetilde{\mathfrak{a}}) + |\mathcal{B}| - \#\{\alpha \in \mathcal{B} \mid \sigma\alpha > 0\} \\
&= \mathrm{codim}_{\widetilde{\mathfrak{u}}} \widetilde{\mathfrak{a}} + \#\{\alpha \in \mathcal{B} \mid \sigma\alpha < 0\} \\
&= \dim \widetilde{\Omega} + \mathcal{F} - \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}} - \vartheta + \mathcal{F} - \widetilde{\mathcal{F}} = \mathcal{F} - \vartheta.
\end{aligned}$$

Therefore $\dim \Omega$ doesn't depend on ξ and is less or equal to $\mathcal{F}$. In other words, Theorem 1.2 holds for all orthogonal subsets of $\widetilde{\Phi}^+$.

4.3. In this Subsection, we consider orthogonal subsets of F_4^+ which don't contain in $\widetilde{\Phi}^+$. In other words, we assume the intersection of D with $\mathcal{B}$ to be non-empty. It's easy to see that if $\beta_1, \beta_2 \in \mathcal{B}$ are orthogonal and $\beta_1 \notin S(\beta_2)$, then $\beta_2 \in S(\beta_1)$, so without loss of generality it can be assumed that $|D \cap \mathcal{B}| = 1$ (see Lemma 2.3). Clearly, D doesn't contain the roots ε_i, $1 \leqslant i \leqslant 4$; in other words, there exists a unique short root contained in D.

The root system F_4 is self-dual, so there exists the bijection $\varphi \colon F_4 \to F_4$ such that $\varphi(F_4^+) = F_4^+$ and $\varphi(S(\alpha)) = S(\varphi(\alpha))$ for a given positive root α. Further, $\varphi(\mathcal{B}) = \{\varepsilon_1 \pm \varepsilon_j, \varepsilon_2 \pm \varepsilon_j, j = 3, 4\}$ (signs

are independent) and $\varphi(\widetilde{\Phi}) \cong C_4$. So if $D \subset \varphi(\widetilde{\Phi}^+)$, then the results of [10] can be applied, and Theorem 1.2 holds for the subset D.

Let $\beta = (\varepsilon_1 + z_2 \cdot \varepsilon_2 + z_3 \cdot \varepsilon_3 + z_4 \cdot \varepsilon_4)/2 \in D$, $z_j = \pm 1$. By previous remarks, $\widetilde{D} = D \setminus \{\beta\}$ coincides either with one of the subsets $\{\varepsilon_1 - z_3 \cdot \varepsilon_3,\ \varepsilon_2 - z_2 z_4 \cdot \varepsilon_4\}$, $\{\varepsilon_1 - z_4 \cdot \varepsilon_3, \varepsilon_2 - z_2 z_3 \cdot \varepsilon_3\}$ or with a one-element subset of them. Since the root $\beta = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$ is fundamental, $\Omega = \xi_\beta e_\beta^* + \widetilde{\Omega}$, and so $\dim \Omega = \dim \widetilde{\Omega}$, where $\widetilde{\Omega} = \Omega_{\widetilde{D},\widetilde{\xi}} \subset \mathfrak{u}^*$, $\widetilde{\xi} = \xi \setminus \{\xi_\beta\}$. On the other hand, r_β acts on the set of positive roots non-equal to β by permutations, so $l(\sigma) - s(\sigma) = (l(\widetilde{\sigma}) + 1) - (s(\widetilde{\sigma}) + 1) = l(\widetilde{\sigma}) - s(\widetilde{\sigma})$, where $\widetilde{\sigma}$ is the involution in W corresponding to $\widetilde{D}$. But Theorem 1.2 holds for the orbit $\widetilde{\Omega}$, so we may assume that $\beta \neq (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$. By the same argument, it can be assumed that the fundamental root $\alpha_1 = \varepsilon_2 - \varepsilon_3$ doesn't belong to D (if $|D| = 2$, then the problem reduces to elementary orbits; if $|D| = 3$, then the problem reduces to orbits associated with two-element subsets).

For a given D, denote by $\mathcal{M} \subset \Phi^+$ a subset satisfying the following conditions. Firstly, if $\alpha + \gamma = \beta \in D$, then $|\mathcal{M} \cap \{\alpha, \gamma\}| = 1$. Secondly, for a given $\gamma \in \mathcal{M}$, there exists $\alpha \in \mathcal{P}$ such that $\alpha + \gamma = \beta \in D$ (here $\mathcal{P} = \Phi^+ \setminus \mathcal{M}$). Thirdly, $(\alpha + \mathcal{M}) \cap D$ consists either of the root β or of the roots $\beta, \widetilde{\beta} = \alpha + \widetilde{\gamma}$, $\widetilde{\gamma} \in \mathcal{M}$, and in the latter case $\widetilde{\gamma} \in S(\beta)$, $(\widetilde{\alpha} + \mathcal{M}) \cap D = \{\beta\}$, where $\widetilde{\alpha} = \beta - \widetilde{\gamma} \in \mathcal{P}$. Assume $\mathcal{M}$ exists. Then $\mathfrak{p} = \sum_{\alpha \in \mathcal{P}} k e_\alpha$ is a maximal isotropic subspace of the canonical form on Ω, so $\dim \Omega = 2 \cdot \operatorname{codim}_{\mathfrak{u}} \mathfrak{p} = 2 \cdot |\mathcal{M}|$ doesn't depend on ξ.

In the table below we list subsets $\mathcal{M}$ for all remaining D (signs $\pm$ in the table are independent). It's straightforward to check that they satisfy the above conditions. We also compute the numbers $\mathcal{F} = l(\sigma) - s(\sigma)$ for all D. One can see that $2 \cdot |\mathcal{M}| \leqslant \mathcal{F}$ for all D. This concludes the proof of Theorem 1.2.

Subset D	Subset $\mathcal{M}$	$\lvert\mathcal{M}\rvert$	$\mathcal{F}$
1) $\varepsilon_1 + \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	ε_1, ε_4, $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 \pm \varepsilon_4$, $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \pm \varepsilon_4)/2$	7	14
2) $\varepsilon_1 - \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	$\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 - \varepsilon_3$, $(\varepsilon_1 \pm \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	4	8
3) $\varepsilon_2 + \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	ε_4, $\varepsilon_2 - \varepsilon_3$	2	4
4) $\varepsilon_1 - \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\varepsilon_1 - \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 \pm \varepsilon_4)/2$	3	6
5) $\varepsilon_1 + \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	ε_1, ε_3, $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 - \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 + \varepsilon_4)/2$	6	12
6) $\varepsilon_2 + \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	ε_3, $\varepsilon_3 \pm \varepsilon_4$	3	6
7) $\varepsilon_2 - \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	$\varepsilon_3 - \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	2	4
8) $\varepsilon_1 - \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	ε_4, $\varepsilon_1 - \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 \pm \varepsilon_4)/2$	4	8
9) $\varepsilon_1 - \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	ε_3, $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 - \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 - \varepsilon_4)/2$	5	10
10) $\varepsilon_2 + \varepsilon_3$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	ε_3, $\varepsilon_3 \pm \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$	4	8
11) $\varepsilon_2 + \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$	ε_4, $\varepsilon_3 + \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2$	3	6
12) $\varepsilon_1 + \varepsilon_3$, $(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$	ε_1, $\varepsilon_2 + \varepsilon_3$, $\varepsilon_1 \pm \varepsilon_4$, $(\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$	8	16

Subset D	Subset $\mathcal{M}$	$\lvert\mathcal{M}\rvert$	$\mathcal{F}$
13) $\varepsilon_1+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	ε_1, $\varepsilon_1-\varepsilon_2$, $\varepsilon_1-\varepsilon_3$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)/2$	7	14
14) $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2\pm\varepsilon_4$	4	8
15) $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2-\varepsilon_4$	3	6
16) $\varepsilon_1+\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_1, ε_4, $\varepsilon_1-\varepsilon_2$, $\varepsilon_1\pm\varepsilon_4$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)/2$	9	18
17) $\varepsilon_1-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_2, $\varepsilon_1-\varepsilon_2$, $\varepsilon_1-\varepsilon_3$, $\varepsilon_2-\varepsilon_3$, $(\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	6	12
18) $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_2, $\varepsilon_2\pm\varepsilon_4$, ε_4, $\varepsilon_2-\varepsilon_3$	5	10
19) $\varepsilon_2-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_2, ε_4, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2+\varepsilon_4$	4	8
20) $\varepsilon_1-\varepsilon_3$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3-\varepsilon_4$, $(\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	6	14
21) $\varepsilon_1+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_1, ε_2, $\varepsilon_1-\varepsilon_3$, $\varepsilon_2\pm\varepsilon_4$, $(\varepsilon_1\pm\varepsilon_2-\varepsilon_3\pm\varepsilon_4)/2$	9	20
22) $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_2, ε_3, $\varepsilon_2\pm\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3-\varepsilon_4$	6	12
23) $\varepsilon_1-\varepsilon_3$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_2, ε_4, $\varepsilon_2\pm\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3\pm\varepsilon_4)/2$	7	16
24) $\varepsilon_1-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_2, ε_3, $\varepsilon_2+\varepsilon_3$, $\varepsilon_3-\varepsilon_4$, $\varepsilon_2\pm\varepsilon_4$, $(\varepsilon_1+\varepsilon_2\pm\varepsilon_3-\varepsilon_4)/2$	8	18
25) $\varepsilon_2-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_2, $\varepsilon_2+\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $\varepsilon_3+\varepsilon_4$, ε_3, ε_4, $\varepsilon_2-\varepsilon_3$	7	14
26) $\varepsilon_1-\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_2, ε_4, $\varepsilon_2+\varepsilon_4$, $\varepsilon_2\pm\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3\pm\varepsilon_4)/2$	7	16
27) $\varepsilon_1-\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3-\varepsilon_4$, $(\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	6	14
28) $\varepsilon_1+\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_1, ε_4, $\varepsilon_1-\varepsilon_2$, $\varepsilon_3\pm\varepsilon_4$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3\pm\varepsilon_4)/2$	7	14
29) $\varepsilon_1-\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $(\varepsilon_1-\varepsilon_2-\varepsilon_3\pm\varepsilon_4)/2$	4	10
30) $\varepsilon_1+\varepsilon_4$, $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)/2$	ε_3, ε_4, $\varepsilon_2+\varepsilon_4$, $\varepsilon_3+\varepsilon_4$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3+\varepsilon_4)/2$	6	14
31) $\varepsilon_1-\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_4, $\varepsilon_2-\varepsilon_3$, $(\varepsilon_1-\varepsilon_2-\varepsilon_3\pm\varepsilon_4)/2$	4	10
32) $\varepsilon_1-\varepsilon_4$, $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$	ε_3, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3-\varepsilon_4$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3-\varepsilon_4)/2$	5	12
33) $\varepsilon_1+\varepsilon_3$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	ε_1, ε_2, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3\pm\varepsilon_4$, $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1-\varepsilon_2+\varepsilon_3\pm\varepsilon_4)/2$	8	18
34) $\varepsilon_1+\varepsilon_4$, $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	ε_1, ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_3+\varepsilon_4$, $\varepsilon_2+\varepsilon_4$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3+\varepsilon_4)/2$	7	16
35) $\varepsilon_1+\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_1, ε_4, $\varepsilon_1-\varepsilon_2$, $\varepsilon_3\pm\varepsilon_4$, $(\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4)/2$	9	20
36) $\varepsilon_1-\varepsilon_4$, $\varepsilon_2+\varepsilon_3$, $(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2$	ε_2, $\varepsilon_2-\varepsilon_3$, $\varepsilon_2-\varepsilon_4$, $\varepsilon_3-\varepsilon_4$, $(\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$	6	14

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